Lecture 1

Monday, January 4, 2021 9:44 AM

1. Intro to SCV.

We identify $\mathbb{R}^{2n} \cong \mathbb{C}^n$ via $z_j = x_j + i y_j$, $j = 1, \dots, n$. Set

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j}\right)$$

$$\mathbb{C} \otimes T_p \mathbb{C}^n = \operatorname{span}_{\mathbb{C}} \left\langle \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_1}, \dots \frac{\partial}{\partial \bar{z}_n} \right\rangle$$

Complex structure: $J : T_p\mathbb{C}^n \to T_p\mathbb{C}^n$, $J^2 = -1$ given by

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \quad J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}.$$

Check: $J\left(\frac{\partial}{\partial z_j}\right) = i \frac{\partial}{\partial z_j}$, $J\left(\frac{\partial}{\partial \bar{z}_j}\right) = -i \frac{\partial}{\partial \bar{z}_j}$

$$T_p^{1,0}\mathbb{C}^n = \operatorname{span}_{\mathbb{C}} \left\langle \frac{\partial}{\partial z_j} \right\rangle_{j=1}^n, \quad T_p^{0,1}\mathbb{C}^n = \operatorname{span}_{\mathbb{C}} \left\langle \frac{\partial}{\partial \bar{z}_j} \right\rangle_{j=1}^n$$

Let $\Omega \subseteq \mathbb{C}^n$ open, connected subset (domain).

Def. A fcn $f : \Omega \to \mathbb{C}$ is holomorphic if $f \in \mathcal{C}^1$ and

$$\frac{\partial f}{\partial \bar{z}_j} = 0, \quad j = 1, \dots, n.$$

Basic facts: ① f is holomorphic $\Rightarrow f \in \mathcal{C}^\infty$.

② f is holomorphic in $\Omega \subseteq \mathbb{C}^n$ and $B(p,r) = \{z : \|z - p\| < r\} \subseteq \Omega$ (open ball) $\Rightarrow$ $f(z) = \sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha (z-p)^\alpha$ with normal convergence (absolute and uniform) in $B(p,r)$.

$(z-p)^\alpha = (z_1 - p_1)^{\alpha_1} \dots (z_n - p_n)^{\alpha_n}$.

$$a_\alpha = \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial z^\alpha}(p), \qquad \frac{\partial^\alpha f}{\partial z^\alpha} = \frac{\partial^{\alpha_1} \dots \partial^{\alpha_n} f}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}, \qquad \alpha! = \alpha_1! \dots \alpha_n!$$

Obs. If $f \in \mathcal{C}^1(\bar{\Omega})$, then for any $Z = \sum c_j \frac{\partial}{\partial z_j}$ tangent to

Obs. If $f \in C^1(\overline{\Omega})$, then for any $Z = \sum c_j \frac{\partial}{\partial z_j}$ tangent to $\partial\Omega$ $(\Omega = \{\rho < 0\})$: $\boxed{\sum_{j=1}^n c_j \frac{\partial \rho}{\partial z_j} = 0}$, $\boxed{\overline{Z} = \sum \overline{c_j} \frac{\partial f}{\partial \overline{z}_j} = 0}$

Def. If $f \in C^1(\partial\Omega)$ and $\overline{Z}f = 0$, for all $Z = \sum_{j=1}^n c_j \frac{\partial}{\partial z_j}$ tangent to $\partial\Omega$, then f is CR function.

Thm (Bochner). If $\Omega \subseteq \mathbb{C}^n$ is bounded, $\partial\Omega$ smooth, $f \in C^1(\partial\Omega)$ CR, then $\exists$ F holomorphic in Ω, $C^1(\overline{\Omega})$, s.t. $F|_{\partial\Omega} = f$.

Remark. We will investigate in this class how this works locally!

Important Observation. Suppose $M = \{\rho = 0\}$, $\tilde{M} = \{\tilde{\rho} = 0\}$ with $\rho = \operatorname{Im} z_n - \varphi(z', \overline{z}', \operatorname{Re} z_n)$, $\tilde{\rho} = \operatorname{Im} z_n - \tilde{\varphi}(z', \overline{z}', \operatorname{Re} z_n)$, $0 \in M, \tilde{M}$

$(z_1, \dots, z_{n-1})$

In general, $\nexists$ biholomorphic map $F: (\mathbb{C}^n, 0) \circlearrowleft$ s.t. $F(\tilde{M}) = M$.

Heuristic argument (due to H. Poincaré): Let us even try to find a formal holomorphic power series $F(z) = \sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha z^\alpha$ that sends $\tilde{M}$ to M, i.e.

$$\rho(F(z), \overline{F(z)}) = 0 \quad \text{when} \quad \tilde{\rho}(z, \overline{z}) = 0.$$

We write $z_j = x_j + iy_j$, $F(z) = (h(z), g(z))$, $h = (h_1, \dots, h_{n-1})$ $\Rightarrow$

$$\operatorname{Im} g(z', x_n + i\tilde{\varphi}(z', \overline{z}', x_n)) = \varphi(h(z', x_n + i\varphi(z', \overline{z}', x_n)), \overline{h}, \operatorname{Re} g)$$

$$\operatorname{Im} g(z', x_n + i\tilde{\varphi}(z', \overline{z'}, x_n)) = \varphi\big(h(z', x_n + i\varphi(z', \overline{z'}, x_n)), \overline{h}, \operatorname{Re} g\big)$$

same arguments as

We expand $\varphi, \tilde{\varphi}$ as follows $\varphi(z', \overline{z'}, x_n) = \sum_{\substack{\alpha,\beta \in \mathbb{Z}_+^{n-1} \\ k \in \mathbb{Z}_+}} \varphi_{\alpha\beta k} (z')^\alpha (\overline{z'})^\beta x_n^k$

and similarly for $\tilde{\varphi}$. If we pick α, β, k and identify the coeffizents of $(z')^\alpha (\overline{z'})^\beta x_n^k$, we get a polynomial equation involving $\varphi_{\alpha\beta k}, \tilde{\varphi}_{\alpha\beta k}$ in which the $a_\gamma, \overline{a_\mu}$ with $|\gamma| = \gamma_1 + \dots + \gamma_n$, $|\mu|$ are bounded by $|\alpha| + |\beta| + k$. (roughly). If we fix $m \in \mathbb{Z}_+$ and consider all such equations w/ $|\alpha| + |\beta| + k \leq m$, then it is easy to convince oneself that the number of equations (multi indices w/ $n-1+n-1+1 = 2n-1$ components) w/ $|\alpha| + |\beta| + k \leq m$ grows faster than the number of available Taylor coeffizents a_γ w/ $|\gamma| \leq m$ (multi indices w/ n components) whenever $n \geq 2$.